др Владимир Балтић

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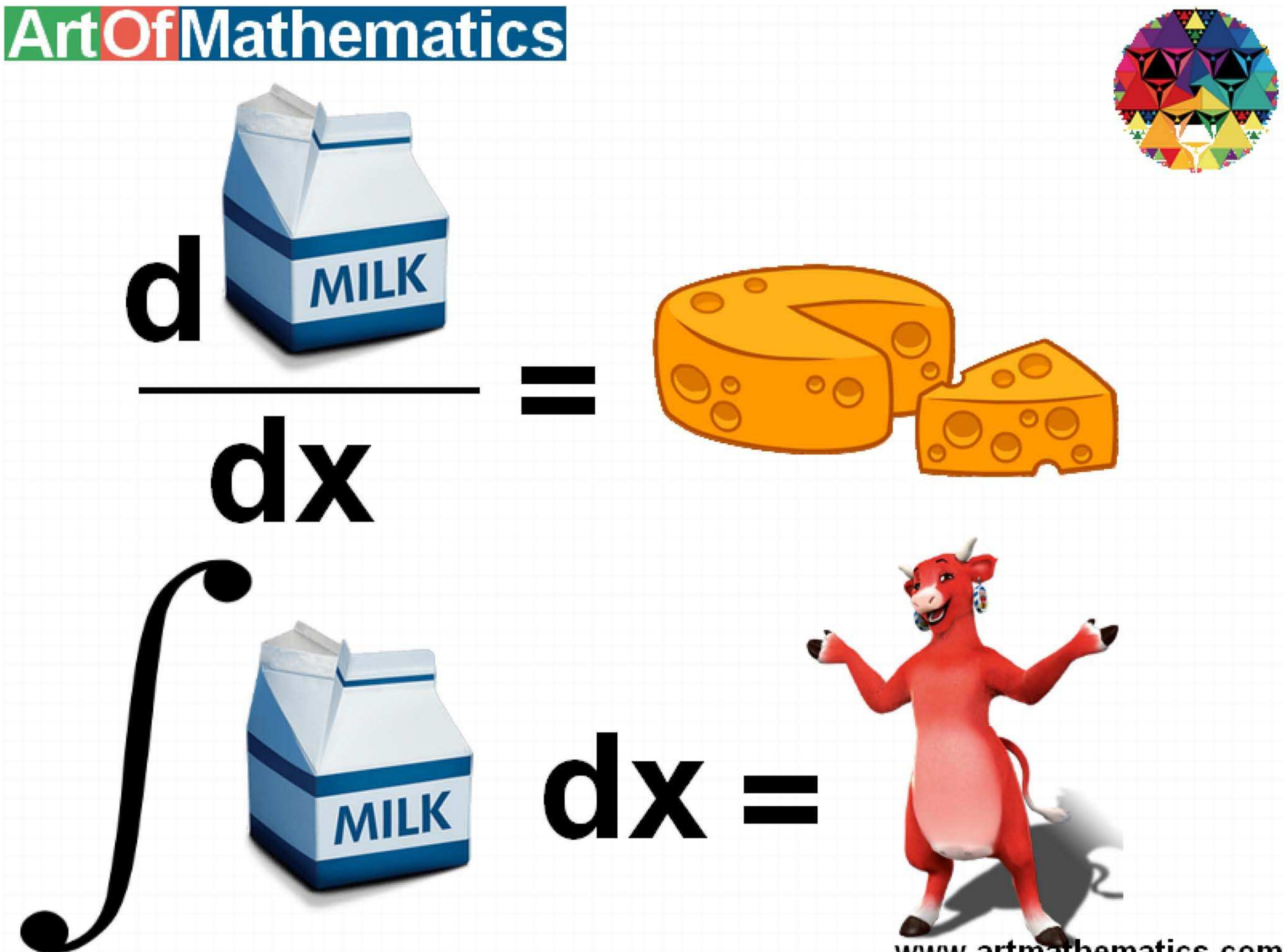
12. термин

Интеграли

Неодређени интеграли

Ако је $F'(x) = f(x)$ $(a < x < b)$, тада је F *примитивна функција* функције f на (a, b). За њу важи да је *неодређени интеграл*

$$\int f(x)\,\mathrm{d}x = F(x) + C.$$


ArtOfMathematics
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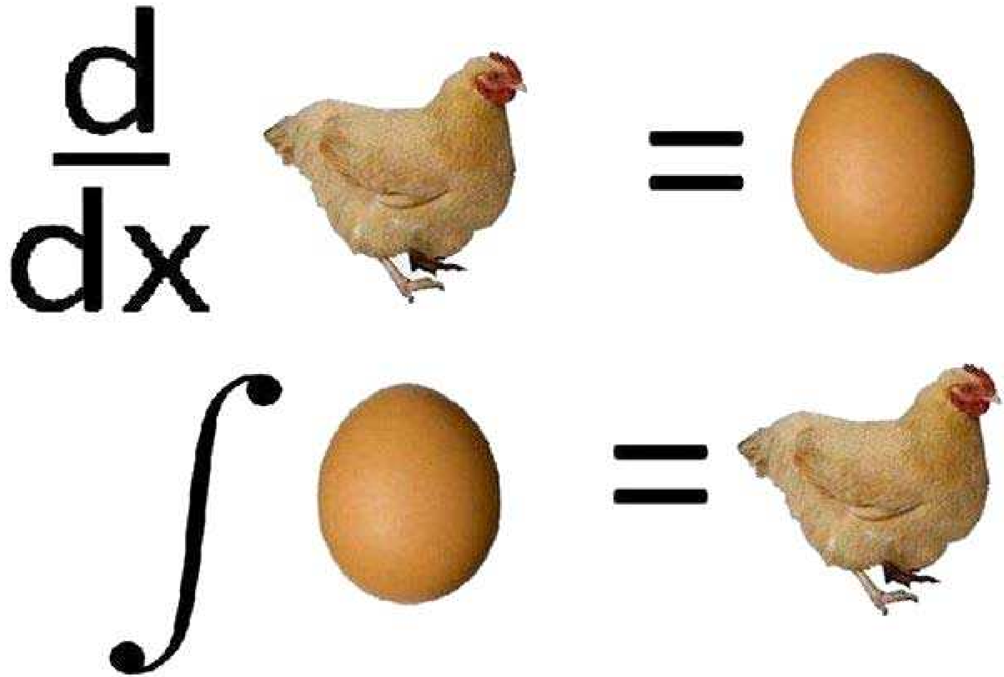
What came first, the derivative or the integral ?



Основна правила.

c, α, β константе, а $f(x)$ и $g(x)$ функције по x.

$$\textbf{1.}\quad \int c \cdot f \,\mathrm{d}x = c \cdot \int f \,\mathrm{d}x,$$

$$\textbf{2.}\quad \int (f \pm g) \,\mathrm{d}x = \int f \,\mathrm{d}x \pm \int g \,\mathrm{d}x$$

$\Rightarrow$ линеарност:

$$\int \big(\alpha f \pm \beta g\big) \,\mathrm{d}x = \alpha \int f \,\mathrm{d}x \pm \beta \int g \,\mathrm{d}x.$$

Таблица интеграла.

1. $\int x^n \, \mathrm{d}x = \frac{x^{n+1}}{n+1} + C$, за $n \neq -1$; **2.** $\int \mathrm{d}x = x + C$;

3. $\int \frac{1}{x} \, \mathrm{d}x = \ln|x| + C$; **4.** $\int \frac{1}{1+x^2} \, \mathrm{d}x = \operatorname{arctg} x + C$;

$$\textbf{5.} \quad \int \frac{1}{1-x^2} \, \mathrm{d}x = \frac{1}{2} \ln \left| \frac{1+x}{1-x} \right| + C;$$

$$\textbf{6.} \quad \int \frac{1}{\sqrt{1-x^2}} \, \mathrm{d}x = \arcsin x + C;$$

$$\textbf{7.} \quad \int \frac{1}{\sqrt{x^2+1}} \, \mathrm{d}x = \ln(x + \sqrt{x^2+1}) + C;$$

$$\textbf{8.} \quad \int \frac{1}{\sqrt{x^2-1}} \, \mathrm{d}x = \ln|x + \sqrt{x^2-1}| + C;$$

9. $\int a^x \, \mathrm{d}x = \frac{a^x}{\ln a} + C$; **10.** $\int e^x \, \mathrm{d}x = e^x + C$;

11. $\int \sin x \, \mathrm{d}x = -\cos x + C$; **12.** $\int \cos x \, \mathrm{d}x = \sin x + C$;

13. $\int \frac{1}{\sin^2 x} \, \mathrm{d}x = -\operatorname{ctg} x + C$; **14.** $\int \frac{1}{\cos^2 x} \, \mathrm{d}x = \operatorname{tg} x + C$.

Одређени интеграли

Њутн–Лајбницова формула:

$$\int_a^b f(x)\,\mathrm{d}x = F(x)\,\Big|_a^b = F(b) - F(a).$$

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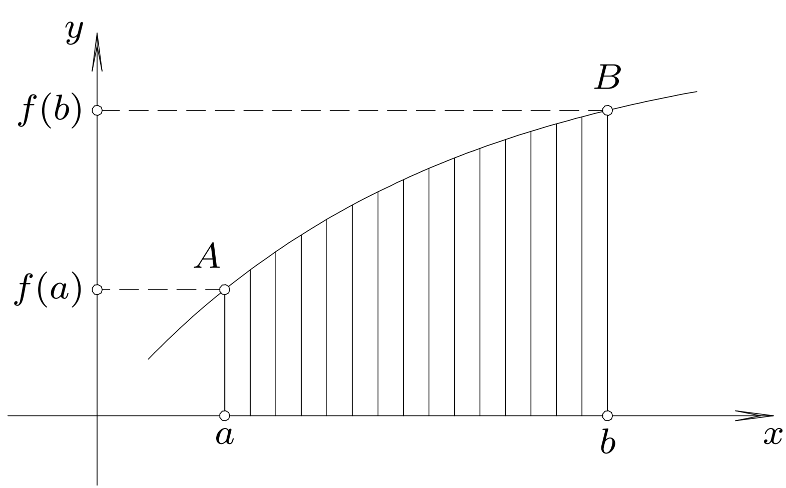
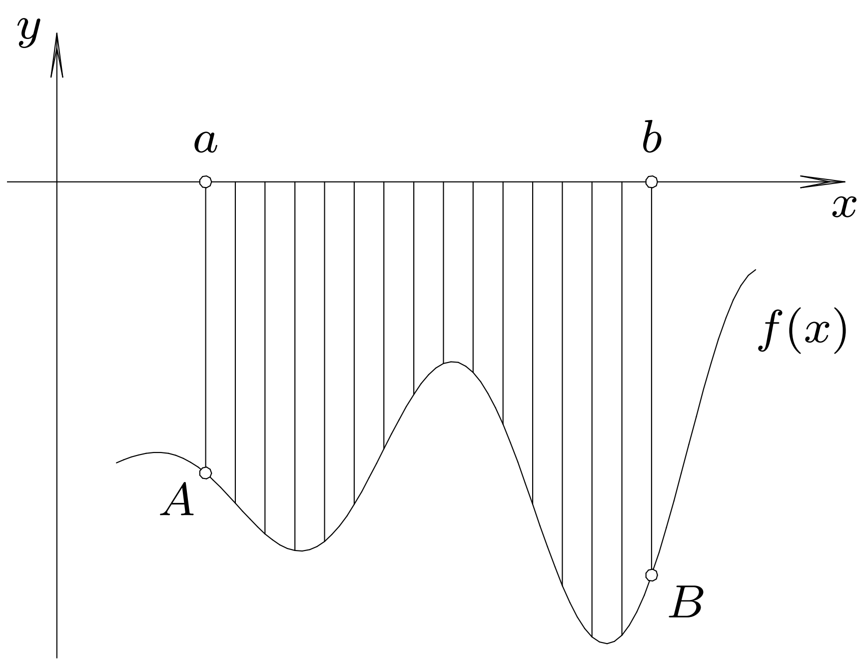
Одређени интеграл представља величину површине испод криве (до x-осе): $P = \displaystyle\int_a^b f(x)\,\mathrm{d}x$.



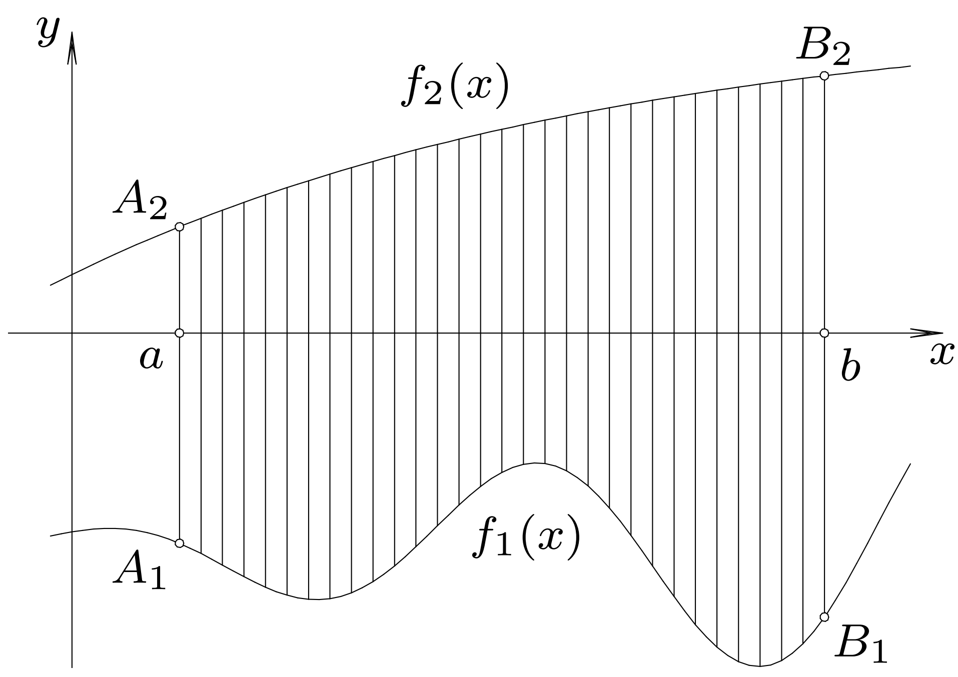
Ако је график испод x-осе онда је:

$$P = -\int_a^b f(x)\,\mathrm{d}x.$$



Најопштији случај је када је ограничена кривама $y = f_1(x)$ и $y = f_2(x)$ ($f_1(x) \leqslant f_2(x)$ на $[a, b]$, тј. $f_1(x)$ је доња крива, а $f_2(x)$ горња крива) и правама $x = a$ и $x = b$ је:

$$P = \int_a^b \big(f_2(x) - f_1(x)\big)\,\mathrm{d}x.$$



Важније особине одређеног интеграла.

1. $\displaystyle\int_a^a f(x)\,\mathrm{d}x = 0;$ **2.** $\displaystyle\int_a^b f(x)\,\mathrm{d}x = -\int_b^a f(x)\,\mathrm{d}x;$

3. адитивност $(a < c < b)$ $\displaystyle\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x,$

4. линеарност $\displaystyle\int_a^b (\alpha f \pm \beta g)\,\mathrm{d}x = \alpha\int_a^b f\,\mathrm{d}x \pm \beta\int_a^b g\,\mathrm{d}x;$

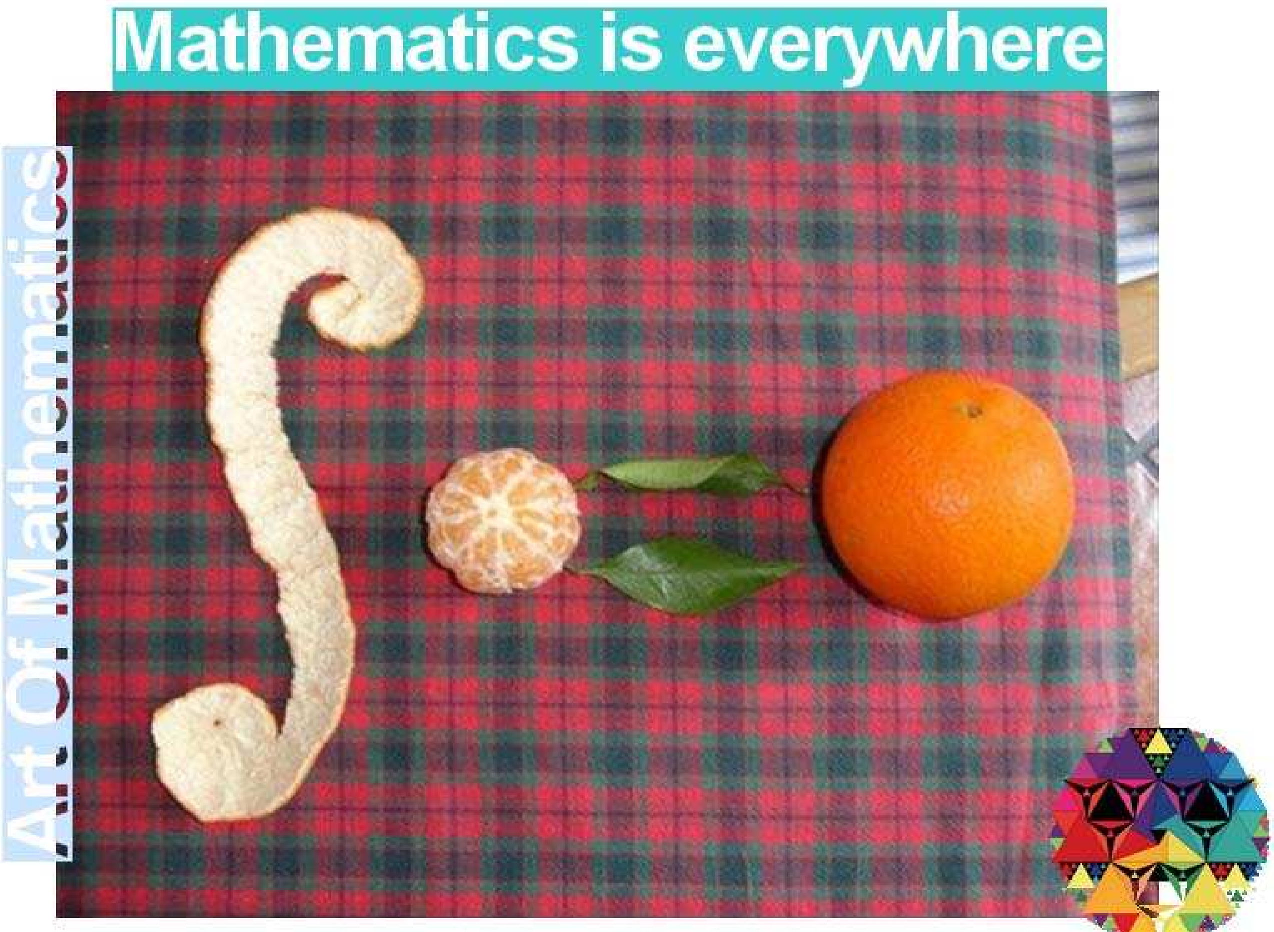
5. $\displaystyle\int\limits_0^a f(x)\,\mathrm{d}x = \int\limits_0^a f(a-x)\,\mathrm{d}x;$

6. непарна ф-ја $\displaystyle\int\limits_{-a}^a f(x)\,\mathrm{d}x = 0;$

7. парна ф-ја $\displaystyle\int\limits_{-a}^a f(x)\,\mathrm{d}x = 2\int\limits_0^a f(x)\,\mathrm{d}x;$

8. периодична ф-ја са периодом T $\displaystyle\int\limits_a^{a+T} f(x)\,\mathrm{d}x = \int\limits_b^{b+T} f(x)\,\mathrm{d}x.$

Уводни задаци



Уводни задаци

1. $\displaystyle\int (2x^2+1)^3\,\mathrm{d}x.$

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Решење. $(a + b)^3 = ???$

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Решење. $\quad (a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$

1. $I = \displaystyle\int (2x^2+1)^3 \,\mathrm{d}x.$

Решење. $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$

$$I = \int (8x^6 + 12x^4 + 6x^2 + 1)\,\mathrm{d}x =$$

$$I = 8\int x^6\,\mathrm{d}x + 12\int x^4\,\mathrm{d}x + 6\int x^2\,\mathrm{d}x + \int \mathrm{d}x =$$

Таблица интеграла.

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$$= \frac{8}{7}x^7 + \frac{12}{5}x^5 + 2x^3 + x + C.$$

■

2. $\displaystyle\int_0^1 (1+\sqrt{x}\,)^2\,\mathrm{d}x.$

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Решење. $\displaystyle I = \int_0^1 (1+x^{1/2})^2\,\mathrm{d}x =$

$$\int\limits_0^1 (1 + 2x^{1/2} + x)\,\mathrm{d}x = \left(x + \frac{4}{3}x^{3/2} + \frac{1}{2}x^2\right)\Big|_0^1 =$$

$$\left(1 + \frac{4}{3}\cdot 1^{3/2} + \frac{1}{2}\cdot 1^2\right) - \left(0 + \frac{4}{3}\cdot 0^{3/2} + \frac{1}{2}\cdot 0^2\right) =$$

$$(1 + \frac{4}{3} + \frac{1}{2}) - (0+0+0) = \frac{6+8+3}{6} - 0 = \frac{17}{6}. \qquad \blacksquare$$

3. $$\int \frac{(x+1)(x^2-3)}{x^3}\,\mathrm{d}x.$$

3. $\displaystyle\int \frac{(x+1)(x^2-3)}{x^3}\,\mathrm{d}x.$

Решење. За интеграле немамо правила за производ и количник!

Па шта да радимо?

3. $I = \displaystyle\int \frac{(x+1)(x^2-3)}{x^3}\,\mathrm{d}x.$

Решење. $$I = \int \frac{x^3 - 3x + x^2 - 3}{x^3}\,\mathrm{d}x =$$

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Решење. $\quad I = \displaystyle\int \frac{x^3 - 3x + x^2 - 3}{x^3}\,\mathrm{d}x =$

$$\int \left(1 - \frac{3}{x^2} + \frac{1}{x} - \frac{3}{x^3}\right)\,\mathrm{d}x =$$

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$$x - 3 \cdot \frac{x^{-1}}{-1} + \frac{x^0}{0} \qquad ???$$

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$$x \; + \; \frac{3}{x} \; + \; \ln|x| \; + \; \frac{3}{2x^2}$$

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$$x - 3 \cdot \tfrac{x^{-1}}{-1} + \ln|x| - 3 \cdot \tfrac{x^{-2}}{-2} =$$

$$x \;+\; \tfrac{3}{x} \;+\; \ln|x| \;+\; \tfrac{3}{2x^2} + C$$

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$$x - 3\cdot\frac{x^{-1}}{-1} + \ln|x| - 3\cdot\frac{x^{-2}}{-2} + C =$$

$$x + \frac{3}{x} + \ln|x| + \frac{3}{2x^2} + C. \qquad \blacksquare$$

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Решење. $$I = \int_0^1 (x \cdot (x \cdot x^{1/2})^{1/2})^{1/2}\,\mathrm{d}x =$$

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Решење. $I = \int_0^1 (x \cdot (x \cdot x^{1/2})^{1/2})^{1/2}\,\mathrm{d}x =$

$\int_0^1 x^{7/8}\,\mathrm{d}x =$

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Решење. $I = \int_0^1 (x \cdot (x \cdot x^{1/2})^{1/2})^{1/2}\,\mathrm{d}x = \int_0^1 x^{7/8}\,\mathrm{d}x = \frac{x^{15/8}}{\frac{15}{8}}\Big|_0^1 = \frac{8}{15}x^{15/8}\Big|_0^1 = \frac{8}{15} - 0 = \frac{8}{15}.$ ∎

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$$I = \lim_{b\to 1^-} \int_0^b \sqrt{\frac{1+x^2}{1-x^4}}\,\mathrm{d}x.$$

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$$I=\lim_{b\to 1^-}\int_0^b \sqrt{\frac{1+x^2}{1-x^4}}\,\mathrm{d}x.$$

$$I=\int_0^1 \sqrt{\frac{1+x^2}{(1-x^2)(1+x^2)}}\,\mathrm{d}x=\int_0^1 \frac{\mathrm{d}x}{\sqrt{1-x^2}}=\arcsin x\,\Big|_0^1=$$

$$\arcsin 1-\arcsin 0=\frac{\pi}{2}-0=\frac{\pi}{2}. \qquad \blacksquare$$

6. $\displaystyle\int \frac{2^{x+1} - 5^{x-1}}{10^x}\,\mathrm{d}x.$

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Решење.

$$I = \int \frac{2^x \cdot 2^1 - 5^x \cdot \frac{1}{5}}{2^x \cdot 5^x}\,\mathrm{d}x = \int \left(2\left(\tfrac{1}{5}\right)^x - \tfrac{1}{5}\left(\tfrac{1}{2}\right)^x\right)\mathrm{d}x =$$

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9. $\int a^x\,\mathrm{d}x = \frac{a^x}{\ln a} + C$; **10.** $\int e^x\,\mathrm{d}x = e^x + C$;

11. $\int \sin x\,\mathrm{d}x = -\cos x + C$; **12.** $\int \cos x\,\mathrm{d}x = \sin x + C$;

13. $\int \frac{1}{\sin^2 x}\,\mathrm{d}x = -\operatorname{ctg} x + C$; **14.** $\int \frac{1}{\cos^2 x}\,\mathrm{d}x = \operatorname{tg} x + C$.

6. $\displaystyle\int \frac{2^{x+1} - 5^{x-1}}{10^x}\,\mathrm{d}x.$

Решење.

$$I = \int \frac{2^x \cdot 2^1 - 5^x \cdot \frac{1}{5}}{2^x \cdot 5^x}\,\mathrm{d}x = \int \left(2\left(\tfrac{1}{5}\right)^x - \tfrac{1}{5}\left(\tfrac{1}{2}\right)^x\right)\mathrm{d}x =$$

$$2 \cdot \frac{\left(\frac{1}{5}\right)^x}{\ln\left(\frac{1}{5}\right)} - \frac{1}{5} \cdot \frac{\left(\frac{1}{2}\right)^x}{\ln\left(\frac{1}{2}\right)} + C. \qquad \blacksquare$$

7. $\displaystyle\int_5^5 e^{-x^2}\,\mathrm{d}x.$

7. $\int_5^5 e^{-x^2}\,\mathrm{d}x$.

Решење.

Постоје неки неодређени интеграли који „не могу да се реше“:

$$\int e^{-x^2}\,\mathrm{d}x, \qquad \int \frac{\sin x}{x}\,\mathrm{d}x\ldots$$

7. $\int_5^5 e^{-x^2}\,\mathrm{d}x.$

Решење.

Постоје неки неодређени интеграли који „не могу да се реше“:

$$\int e^{-x^2}\,\mathrm{d}x, \qquad \int \frac{\sin x}{x}\,\mathrm{d}x\ldots$$

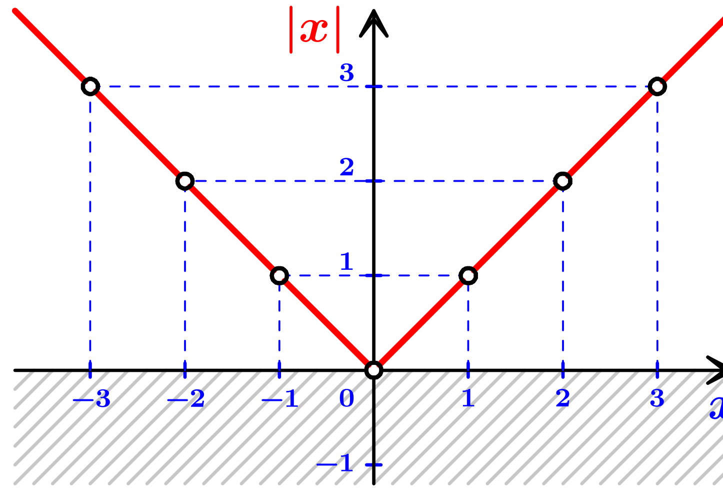
$$I = \int_5^5 e^{-x^2}\,\mathrm{d}x = F(x)\,\Big|_5^5 = F(5) - F(5) = 0. \qquad \blacksquare$$

8. $\displaystyle\int_{-4}^{2} |x| \, \mathrm{d}x.$

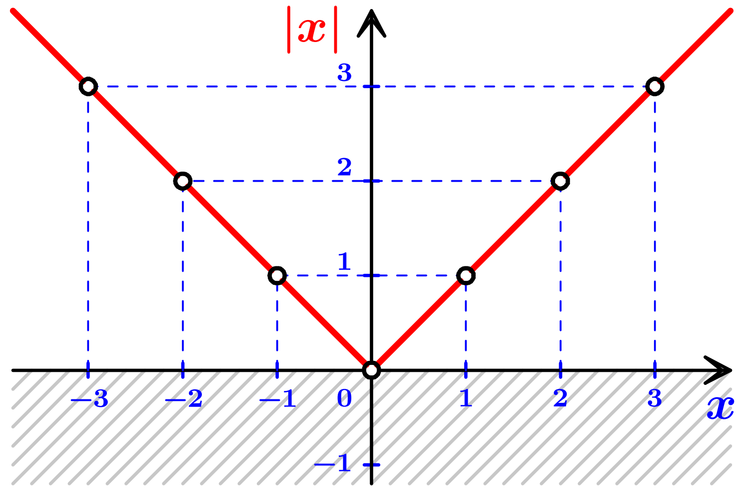
8. $I = \int_{-4}^{2} |x| \, \mathrm{d}x.$

Решење 1.

$$|x| = \begin{cases} x, & x \geqslant 0 \\ -x, & x < 0 \end{cases}$$



8. $I = \int_{-4}^{2} |x| \, \mathrm{d}x.$

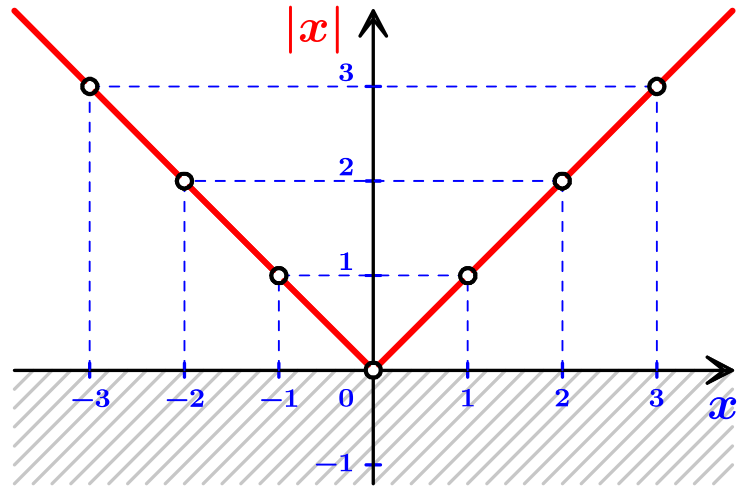


Решење 1.

$$|x| = \begin{cases} x, & x \geqslant 0 \\ -x, & x < 0 \end{cases}$$

$$I = \int_{-4}^{0} |x| \, \mathrm{d}x + \int_{0}^{2} |x| \, \mathrm{d}x = \int_{-4}^{0} (-x) \, \mathrm{d}x + \int_{0}^{2} x \, \mathrm{d}x$$

8. $I=\displaystyle\int_{-4}^{2}|x|\,\mathrm{d}x.$



Решење 1.

$$|x|=\begin{cases} x, & x\geqslant 0\\ -x, & x<0\end{cases}$$

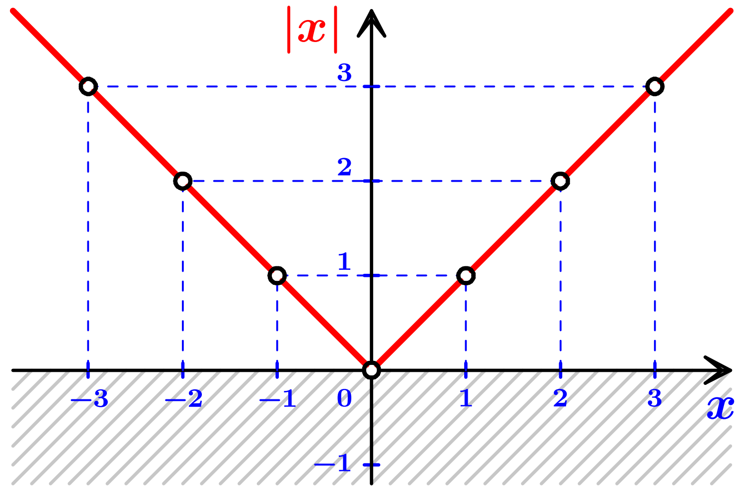
$$I=\int_{-4}^{0}|x|\,\mathrm{d}x+\int_{0}^{2}|x|\,\mathrm{d}x=\int_{-4}^{0}(-x)\,\mathrm{d}x+\int_{0}^{2}x\,\mathrm{d}x$$

$$I=-\frac{x^2}{2}\,\Big|_{-4}^{0}+\frac{x^2}{2}\,\Big|_{0}^{2}=-\frac{0^2}{2}-\left(-\frac{(-4)^2}{2}\right)+\frac{2^2}{2}-\frac{0^2}{2}$$

$$I=8+2=10.$$

■

8. $I = \int_{-4}^{2} |x| \, \mathrm{d}x.$



Решење 1.

$$|x| = \begin{cases} x, & x \geqslant 0 \\ -x, & x < 0 \end{cases}$$

$$I = \int_{-4}^{0} |x| \, \mathrm{d}x + \int_{0}^{2} |x| \, \mathrm{d}x = \int_{-4}^{0} (-x) \, \mathrm{d}x + \int_{0}^{2} x \, \mathrm{d}x$$

$$I = -\frac{x^2}{2} \Bigg|_{-4}^{0} + \frac{x^2}{2} \Bigg|_{0}^{2} = -\frac{0^2}{2} - \left(-\frac{(-4)^2}{2} \right) + \frac{2^2}{2} - \frac{0^2}{2}$$

$I = 8 + 2 = 10.$ ■

Решење 2.

Преко површина $\triangle$... ■

Смена променљиве

неодређени инт.

$\boxed{\varphi(x) = t}$, тада је $\varphi'(x)\,\mathrm{d}x = \,\mathrm{d}t$.

Нека су функције f, φ и φ' непрекидне:

$$\int f(\varphi(x))\varphi'(x)\,\mathrm{d}x = \int f(t)\,\mathrm{d}t + C.$$

одређени инт.

$\boxed{\varphi(x) = t}$, ако важи:

$f(\varphi)$ је непрекидна на $[a, b]$, $\varphi(a) = \alpha$, $\varphi(b) = \beta$,

φ је монотона ($\nearrow$ или $\searrow$) на $[a, b]$:

$$\int_a^b f(\varphi(x))\varphi'(x)\,\mathrm{d}x = \int_\alpha^\beta f(t)\,\mathrm{d}t.$$

9. $\displaystyle \int \frac{\mathrm{d}x}{x+5}\,.$

9. $I = \displaystyle\int \frac{\mathrm{d}x}{x+5}\,.$

Решење.

[смена $\boxed{t = x + 5}$, $1 \cdot \mathrm{d}t = 1 \cdot \mathrm{d}x,\ \ \mathrm{d}t = \mathrm{d}x,$]

9. $I = \displaystyle\int \frac{\mathrm{d}x}{x+5}$.

Решење.

[смена $\boxed{t = x + 5}$, $1 \cdot \mathrm{d}t = 1 \cdot \mathrm{d}x$, $\mathrm{d}t = \mathrm{d}x$,]

$I = \ln|t| + C$.

9. $I = \int \frac{\mathrm{d}x}{x+5}\,.$

Решење.

$[$смена $\boxed{t = x + 5}$, $1 \cdot \mathrm{d}t = 1 \cdot \mathrm{d}x,\ \ \mathrm{d}t = \mathrm{d}x,]$

$I = \ln|t| + C = \ln|x+5| + C.$ ■

10. $\displaystyle\int xe^{-x^2}\,\mathrm{d}x$.

10. $\displaystyle\int xe^{-x^2}\,\mathrm{d}x$.

Результат. $\displaystyle I = -\frac{1}{2}e^{-x^2} + C.$ ■

11. $\displaystyle\int \frac{x\,\mathrm{d}x}{1+x^2}\,.$!!!

11. $\displaystyle\int \frac{x\,\mathrm{d}x}{1+x^2}\,.$!!!

Результат. $$I = \frac{1}{2}\ln(1+x^2) + C.$$ ■

12. $\displaystyle\int \sin^5 x \cos x \, \mathrm{d}x.$

12. $\displaystyle\int \sin^5 x \cos x \, \mathrm{d}x.$

Результат. $\qquad I = \dfrac{1}{6}\sin^6 x + C.$ ■

13. $\int \sqrt{e^x + 1} \cdot e^x \,\mathrm{d}x.$

13. $\displaystyle\int \sqrt{e^x + 1} \cdot e^x \,\mathrm{d}x.$

Результат. $$I = \frac{2\left(\sqrt{e^x + 1}\,\right)^3}{3} + C. \qquad \blacksquare$$

14. $\displaystyle\int \operatorname{tg} x \, \mathrm{d}x.$

14. $\displaystyle\int \operatorname{tg} x \, \mathrm{d}x.$

Решење. $\displaystyle I = \int \operatorname{tg} x \, \mathrm{d}x = \int \frac{\sin x}{\cos x} \, \mathrm{d}x =$

14. $\displaystyle\int \operatorname{tg} x \,\mathrm{d}x.$

Решење. $\displaystyle I = \int \operatorname{tg} x \,\mathrm{d}x = \int \frac{\sin x}{\cos x}\,\mathrm{d}x =$

[смена $\boxed{t = \cos x}$, $\mathrm{d}t = -\sin x\,\mathrm{d}x,\ \sin x\,\mathrm{d}x = -\mathrm{d}t$]

$$I = \int \frac{-\mathrm{d}t}{t} = -\ln|t| + C = -\ln|\cos x| + C. \qquad \blacksquare$$

15. $$\int \frac{\mathrm{d}x}{x\sqrt{x^2+1}}.$$

16. $\displaystyle\int_1^{e^3} \frac{\ln x}{x}\, \mathrm{d}x.$

16. $I = \displaystyle\int_1^{e^3} \frac{\ln x}{x}\,\mathrm{d}x.$

Решење.

$$I = \left[\text{смена } \boxed{t = \ln x},\ \mathrm{d}t = \frac{1}{x}\,\mathrm{d}x;\ \begin{array}{c|c} x & t \\ \hline e^3 & \ln e^3 = 3 \\ 1 & \ln 1 = 0 \end{array}\right] = \int_0^3 t\,\mathrm{d}t = \frac{t^2}{2}\,\Big|_0^3 = \frac{9}{2} - 0 = \frac{9}{2}.$$

$\blacksquare$

17. $\displaystyle\int_0^1 (4x - e^{2x})\,\mathrm{d}x.$

17. $\displaystyle\int_0^1 (4x - e^{2x})\,\mathrm{d}x.$

Решење.

$$I = \int_0^1 (4x - e^{2x})\,\mathrm{d}x = \int_0^1 4x\,\mathrm{d}x - \int_0^1 e^{2x}\,\mathrm{d}x$$

17. $\displaystyle\int_0^1 (4x - e^{2x})\,\mathrm{d}x.$

Решење.

$$I = \int_0^1 (4x - e^{2x})\,\mathrm{d}x = \int_0^1 4x\,\mathrm{d}x - \int_0^1 e^{2x}\,\mathrm{d}x$$

$$I_1 = \int\limits_0^1 4x\,\mathrm{d}x = 4 \cdot \frac{x^2}{2}\,\Big|_0^1 = 2 - 0 = 2$$

17. $\displaystyle\int_0^1 (4x - e^{2x})\,\mathrm{d}x.$

Решење.

$$I = \int_0^1 (4x - e^{2x})\,\mathrm{d}x = \int_0^1 4x\,\mathrm{d}x - \int_0^1 e^{2x}\,\mathrm{d}x$$

$$I_1 = \int\limits_0^1 4x\,\mathrm{d}x = 4 \cdot \frac{x^2}{2}\,\Big|_0^1 = 2 - 0 = 2$$

$$I_2 = \int\limits_0^1 e^{2x}\,\mathrm{d}x = \left[\text{смена } \boxed{t = 2x},\ \mathrm{d}t = 2\,\mathrm{d}x,\ \tfrac{1}{2}\,\mathrm{d}t = \mathrm{d}x;\ \begin{array}{c|c} x & t \\ \hline 1 & 2 \\ \hline 0 & 0 \end{array}\right] = \frac{1}{2}\int\limits_0^2 e^t\,\mathrm{d}t = \frac{1}{2}e^t\,\Big|_0^2 = \frac{1}{2}e^2 - \frac{1}{2}$$

17. $\displaystyle\int_0^1 (4x - e^{2x})\,\mathrm{d}x.$

Решење.

$$I = \int_0^1 (4x - e^{2x})\,\mathrm{d}x = \int_0^1 4x\,\mathrm{d}x - \int_0^1 e^{2x}\,\mathrm{d}x$$

$$I_1 = \int\limits_0^1 4x\,\mathrm{d}x = 4\cdot\frac{x^2}{2}\,\Big|_0^1 = 2 - 0 = 2$$

$$I_2 = \int\limits_0^1 e^{2x}\,\mathrm{d}x = \left[\text{смена}\ \boxed{t = 2x},\ 2\,\mathrm{d}x = \mathrm{d}t,\ \mathrm{d}x = \tfrac{1}{2}\,\mathrm{d}t;\ \begin{array}{c|c} x & t \\ \hline 1 & 2 \\ \hline 0 & 0 \end{array}\right] = \frac{1}{2}\int\limits_0^2 e^t\,\mathrm{d}t = \frac{1}{2}e^t\,\Big|_0^2 = \frac{1}{2}e^2 - \frac{1}{2}$$

$$I = I_1 - I_2 = 2 - \left(\tfrac{1}{2}e^2 - \tfrac{1}{2}\right) = \tfrac{5}{2} - \tfrac{1}{2}e^2. \qquad \blacksquare$$

18. $\displaystyle\int_0^1 e^x + e^x \,\mathrm{d}x.$

18. $\int_0^1 e^{x+e^x}\,\mathrm{d}x.$

Решење. $$I = \int_0^1 e^x \cdot e^{e^x}\,\mathrm{d}x$$

18. $\displaystyle\int_0^1 e^{x+e^x}\,\mathrm{d}x.$

Решење. $\displaystyle I=\int_0^1 e^x\cdot e^{e^x}\,\mathrm{d}x$

x	t
1	$e^1=e$
0	$e^0=1$

смена: $\boxed{t=e^x}$, $\mathrm{d}t=e^x\,\mathrm{d}x$;

$\displaystyle I=\int_1^e e^t\,\mathrm{d}t=$

18. $\displaystyle\int_0^1 e^{x + e^x}\,\mathrm{d}x.$

Решење. $\displaystyle I = \int_0^1 e^x \cdot e^{e^x}\,\mathrm{d}x$

x	t
1	$e^1 = e$
0	$e^0 = 1$

смена: $\boxed{t = e^x}$, $\mathrm{d}t = e^x\,\mathrm{d}x;$

$$I = \int_1^e e^t\,\mathrm{d}t = e^t\,\Big|_1^e = e^e - e^1 = e^e - e. \qquad \blacksquare$$

Парцијална интеграција

неодређени инт.

$$\int u(x)\,\mathrm{d}v(x) = u(x)v(x) - \int v(x)\,\mathrm{d}u(x) + C$$

(чешће пишемо $\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u$).

одређени инт.

$$\int_a^b u(x)\,\mathrm{d}v(x) = u(x)v(x)\,\Big|_a^b - \int_a^b v(x)\,\mathrm{d}u(x) =$$

$$u(b)v(b) - u(a)v(a) - \int_a^b v(x)\,\mathrm{d}u(x).$$

Парцијална интеграција

неодређени инт.

$$\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u.$$

одређени инт.

$$\int_a^b u\,\mathrm{d}v = uv\Big|_a^b - \int_a^b v\,\mathrm{d}u.$$

Парцијална интеграција

неодређени инт. $$\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u.$$

одређени инт. $$\int_a^b u\,\mathrm{d}v = uv\Big|_a^b - \int_a^b v\,\mathrm{d}u.$$

$$\begin{array}{cccc} & u & \mathrm{d}v & \\ \text{извод} & \downarrow & \downarrow & \text{инт.} \\ & \mathrm{d}u & v & \end{array}$$

$$v = \int \mathrm{d}v$$

Парцијална интеграција

неодређени инт. $\displaystyle\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u.$

одређени инт. $\displaystyle\int_a^b u\,\mathrm{d}v = uv\Big|_a^b - \int_a^b v\,\mathrm{d}u.$

$$\begin{array}{cccc} & u & \mathrm{d}v & \\ \text{извод} & \downarrow & \downarrow & \text{инт.} \\ & \mathrm{d}u & v & \\ & & \multicolumn{2}{l}{v = \int \mathrm{d}v} \end{array}$$

$$\begin{array}{cccccc} u: & L & A & X & S & E \\ & \ln & \mathrm{arctg} & x^n & \sin & e^x \\ & & \arcsin & \text{пол} & \cos & \end{array}$$

19. $\displaystyle\int \operatorname{arctg} x \, \mathrm{d}x$.

19. $\displaystyle\int \operatorname{arctg} x \,\mathrm{d}x.$

Решење.

парцијална:

$$\boxed{\begin{array}{cc} u = \operatorname{arctg} x & \mathrm{d}v = \mathrm{d}x \\ \downarrow & \downarrow \\ \mathrm{d}u = \dfrac{\mathrm{d}x}{1+x^2} & v = x \end{array}}$$

19. $\displaystyle\int \operatorname{arctg} x \,\mathrm{d}x.$

Решење.

парцијална:

$$\boxed{\begin{array}{ccc} u = \operatorname{arctg} x & & \mathrm{d}v = \mathrm{d}x \\ \downarrow & & \downarrow \\ \mathrm{d}u = \dfrac{\mathrm{d}x}{1+x^2} & & v = x \end{array}}$$

$$I = x \cdot \operatorname{arctg} x - \int \frac{x \,\mathrm{d}x}{1+x^2}$$

19. $\displaystyle\int \operatorname{arctg} x \, \mathrm{d}x.$

Решење.

парцијална:

$$\boxed{\begin{array}{cc} u = \operatorname{arctg} x & \mathrm{d}v = \mathrm{d}x \\ \downarrow & \downarrow \\ \mathrm{d}u = \dfrac{\mathrm{d}x}{1+x^2} & v = x \end{array}}$$

$$I = x \cdot \operatorname{arctg} x - \int \frac{x \, \mathrm{d}x}{1+x^2} \qquad \text{!!! (то 11. зад)}$$

$$I = x \operatorname{arctg} x - \frac{1}{2} \ln(1+x^2) + C. \qquad \blacksquare$$

20. $\displaystyle\int \ln x \, \mathrm{d}x.$

20. $\displaystyle\int \ln x \,\mathrm{d}x.$

Решење.

$$\boxed{\begin{array}{ccc} u = \ln x & & \mathrm{d}v = \mathrm{d}x \\ \downarrow & & \downarrow \\ \mathrm{d}u = \dfrac{1}{x}\,\mathrm{d}x & & v = x \end{array}}$$

парцијална:

20. $\displaystyle\int \ln x \,\mathrm{d}x.$

Решење.

парцијална:

$$\boxed{\begin{array}{cc} u = \ln x & \mathrm{d}v = \mathrm{d}x \\ \downarrow & \downarrow \\ \mathrm{d}u = \dfrac{1}{x}\,\mathrm{d}x & v = x \end{array}}$$

$$I = x \cdot \ln x - \int x \cdot \frac{1}{x}\,\mathrm{d}x =$$

20. $\int \ln x \,\mathrm{d}x.$

$$\boxed{\begin{array}{cc} u = \ln x & \mathrm{d}v = \mathrm{d}x \\ \downarrow & \downarrow \\ \mathrm{d}u = \dfrac{1}{x}\,\mathrm{d}x & v = x \end{array}}$$

Решење.

парцијална:

$$I = x \cdot \ln x - \int x \cdot \frac{1}{x}\,\mathrm{d}x = x \cdot \ln x - \int \mathrm{d}x =$$

$$I = x \cdot \ln x - x + C.$$

■

21. $$\int \frac{x^2 - x + 3}{e^{2x}}\, \mathrm{d}x.$$

21. $\displaystyle\int \frac{x^2 - x + 3}{e^{2x}}\,\mathrm{d}x.$

Решење. $\displaystyle I = \int (x^2 - x + 3)e^{-2x}\,\mathrm{d}x = \ldots$

21. $\displaystyle\int \frac{x^2 - x + 3}{e^{2x}}\,\mathrm{d}x.$

Решење. $\displaystyle I = \int (x^2 - x + 3)e^{-2x}\,\mathrm{d}x = \ldots = -\frac{1}{2}(x^2 + 3)e^{-2x} + C.$ ■

22. $\displaystyle\int_0^1 x^3 e^x \, \mathrm{d}x.$

22. $\displaystyle\int_0^1 x^3 e^x \, \mathrm{d}x.$

Решење. 3 парцијалне:

$$I = x^3 e^x \Big|_0^1 - \int\limits_0^1 3x^2 e^x \, \mathrm{d}x =$$

$$(1^3 \cdot e^1) - (0^3 \cdot e^0) - 3 \cdot \left[x^2 e^x \Big|_0^1 - \int\limits_0^1 2x e^x \, \mathrm{d}x \right] =$$

$$e - 3e + 6 \int\limits_0^1 x e^x \, \mathrm{d}x = -2e + 6 \left[x e^x \Big|_0^1 - \int\limits_0^1 e^x \, \mathrm{d}x \right] =$$

$$-2e + 6e - 6e^x \Big|_0^1 = 4e - 6(e-1) = 6 - 2e. \qquad \blacksquare$$

23. $\displaystyle\int x^2 \sin 2x \, \mathrm{d}x.$

за домаћи!

24. $\displaystyle\int e^{\sqrt{x}}\,\mathrm{d}x.$

24. $\displaystyle\int e^{\sqrt{x}}\,\mathrm{d}x.$

Решење. Прво смена:

$I = \left[\text{смена } \boxed{t = \sqrt{x}},\ x = t^2,\ \mathrm{d}x = 2t\,\mathrm{d}t\right] = 2\displaystyle\int e^t t\,\mathrm{d}t,$

24. $\displaystyle\int e^{\sqrt{x}}\,\mathrm{d}x.$

Решење. Прво смена:

$I = [\text{смена } \boxed{t = \sqrt{x}},\ x = t^2,\ \mathrm{d}x = 2t\,\mathrm{d}t] = 2\displaystyle\int e^t t\,\mathrm{d}t,$ а онда парцијалном долазимо до:

$I = 2(t-1)e^t + C = 2(\sqrt{x}-1)e^{\sqrt{x}} + C.$ ■

25. $\displaystyle\int e^{2x} \sin 3x \,\mathrm{d}x.$

25. $\displaystyle\int e^{2x}\sin 3x\,\mathrm{d}x.$

Результат. $$I = \frac{e^{2x}}{13}(2\sin 3x - 3\cos 3x) + C.$$ ■

26. $\displaystyle\int \frac{2x \operatorname{arctg} x^2}{1+x^4}\,\mathrm{d}x.$

за домаћи!

27. $\displaystyle\int_{-1}^{1} \sqrt{1-x^2}\,\mathrm{d}x.$

за домаћи!

КРАЈ ЧАСА